Resit exam (online) — Functional Analysis (WIFA–08)

Tuesday 7 July 2020, 8.30h–11.30h CEST (plus 30 minutes for uploading)

University of Groningen

Instructions

1. Only references to the lecture notes and slides are allowed. References to other sources are *not* allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
4. Write both your name and student number on the answer sheets!
5. This exam comes in two versions. Both versions consist of five problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
6. Please submit your work as a single PDF file.

Version 1 (for odd student numbers)

Problem 1 (5 + 10 + 10 = 25 points)

Consider the following linear space:

$$\mathcal{W} = \left\{ x = (x_1, x_2, x_3, \dots) \, : \, x_k \in \mathbb{K}, \quad \sup_{k \in \mathbb{N}} |x_k| w_k < \infty \right\},$$

where $w_k \geq 1$ for all $k \in \mathbb{N}$.

(a) Show that $\mathcal{W} \subset \ell^\infty$.

(b) Prove that $\|x\|_\mathcal{W} = \sup_{k \in \mathbb{N}} |x_k| w_k$ is a norm on $\mathcal{W}$.

(c) Let $w_k = k$. Is the norm $\|\cdot\|_\mathcal{W}$ equivalent to the sup-norm $\|\cdot\|_\infty$?

Problem 2 (5 + 10 + 15 = 30 points)

Equip the space $\mathcal{C}([0,1], \mathbb{K})$ with the norm $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$, and consider the following linear operator:

$$T : \mathcal{C}([0,1], \mathbb{K}) \to \mathcal{C}([0,1], \mathbb{K}), \quad Tf(x) = f(1-x).$$

(a) Compute the operator norm of T.

(b) Show that $\lambda = 1$ and $\lambda = -1$ are eigenvalues of T.

(c) Prove that if $\lambda \notin \{-1, 1\}$, then $\lambda \in \rho(T)$ by explicitly computing $(T - \lambda)^{-1}$.
Hint: in the equation $f(1-x) - \lambda f(x) = g(x)$ replace x by $1-x$ to get a system of two equations from which $f(x)$ can be solved.

Problem 3 (5 + 5 + 5 = 15 points)

Let X be a Hilbert space over $\mathbb{C}$, and let $T : X \to X$ be a linear operator such that

$$(Tx, y) = (x, Ty) \quad \text{for all} \quad x, y \in X.$$

Prove the following statements:

(a) For $y \neq 0$ the map $f_y : X \to \mathbb{C}$ defined by $f_y(x) = (Tx, y)/\|y\|$ is linear;

(b) $\sup_{y \neq 0} |f_y(x)| < \infty$ for all $x \in X$;

(c) T is bounded.

Problem 4 (8 points)

Consider the following linear operator:

$$T : \ell^\infty \to \ell^\infty, \quad T(x_1, x_2, x_3, x_4, \dots) = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \dots).$$

Prove that $\operatorname{ran} T$ is *not* closed in ℓ^∞.

Problem 5 (4 + 4 + 4 = 12 points)

On the linear space $\mathbb{R}^2$ we take the following norm:

$$\|x\|_1 = |x_1| + |x_2|, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

(a) Show that norm of the linear map

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = a_1 x_1 + a_2 x_2,$$

is given by $\|f\| = \max\{|a_1|, |a_2|\}$.

(b) Let $V = \operatorname{span}\{(1, 0)\}$ and $g(x) = 7x_1 + 5x_2$. Compute all $a_1, a_2 \in \mathbb{R}$ such that:

(i) $f(x) = g(x)$ for all $x \in V$;

(ii) $\|f\| = \|g\|$.

(c) Discuss the implication of part (b) for the Hahn-Banach Theorem.

End of test ("version 1", 90 points)

Version 2 (for even student numbers)

Problem 1 (5 + 10 + 10 = 25 points)

Consider the following linear space:

$$\mathcal{W} = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K}, \quad \sup_{k\in\mathbb{N}} |x_k| w_k < \infty \right\},$$

where $w_k \geq 1$ for all $k \in \mathbb{N}$.

(a) Show that $\mathcal{W} \subset \ell^\infty$.

(b) Prove that $\|x\|_{\mathcal{W}} = \sup_{k\in\mathbb{N}} |x_k| w_k$ is a norm on $\mathcal{W}$.

(c) Let $w_k = (k+1)/k$. Is the norm $\|\cdot\|_{\mathcal{W}}$ equivalent to the sup-norm $\|\cdot\|_\infty$?

Problem 2 (5 + 10 + 15 = 30 points)

Equip the space $\mathcal{C}([-\pi,\pi],\mathbb{K})$ with the norm $\|f\|_\infty = \sup_{x\in[-\pi,\pi]} |f(x)|$, and consider the following linear operator:

$$T : \mathcal{C}([-\pi,\pi],\mathbb{K}) \to \mathcal{C}([-\pi,\pi],\mathbb{K}), \quad Tf(x) = f(-x).$$

(a) Compute the operator norm of T.

(b) Show that $\lambda = 1$ and $\lambda = -1$ are eigenvalues of T.

(c) Prove that if $\lambda \notin \{-1, 1\}$, then $\lambda \in \rho(T)$ by explicitly computing $(T-\lambda)^{-1}$.
Hint: in the equation $f(-x) - \lambda f(x) = g(x)$ replace x by $-x$ to get a system of two equations from which $f(x)$ can be solved.

Problem 3 (5 + 5 + 5 = 15 points)

Let X be a Hilbert space over $\mathbb{C}$, and let $T : X \to X$ be a linear operator such that

$$(Tx, y) = (x, Ty) \quad \text{for all} \quad x, y \in X.$$

Prove the following statements:

(a) For $y \neq 0$ the map $f_y : X \to \mathbb{C}$ defined by $f_y(x) = (Tx, y)/\|y\|$ is linear;

(b) $\sup_{y\neq 0} |f_y(x)| < \infty$ for all $x \in X$;

(c) T is bounded.

Problem 4 (8 points)

Consider the following linear operator:

$$T : \ell^1 \to \ell^1, \quad T(x_1, x_2, x_3, x_4, \dots) = (x_1, \tfrac{1}{4}x_2, \tfrac{1}{9}x_3, \tfrac{1}{16}x_4, \dots).$$

Prove that $\operatorname{ran} T$ is *not* closed in ℓ^1.

Problem 5 (4 + 4 + 4 = 12 points)

On the linear space $\mathbb{R}^2$ we take the following norm:

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

(a) Show that norm of the linear map

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = a_1 x_1 + a_2 x_2,$$

is given by $\|f\| = |a_1| + |a_2|$.

(b) Let $V = \operatorname{span}\{(1,1)\}$ and $g(x) = 3x_1 + 5x_2$. Compute all $a_1, a_2 \in \mathbb{R}$ such that:

 (i) $f(x) = g(x)$ for all $x \in V$;

 (ii) $\|f\| = \|g\|$.

(c) Discuss the implication of part (b) for the Hahn-Banach Theorem.

End of test ("version 2", 90 points)

Solution of problem 1, version 1 and 2 (5 + 10 + 10 = 25 points)

(a) We have that $|x_k| \leq |x_k| w_k$ for all $k \in \mathbb{N}$. Taking the supremum over all $k \in \mathbb{K}$ gives

$$\sup_{k\in\mathbb{N}} |x_k| \leq \sup_{k\in\mathbb{N}} |x_k| w_k.$$

Hence, if $x \in \mathcal{W}$, then $\sup_{k\in\mathbb{N}} |x_k| < \infty$ which means that $x \in \ell^\infty$.
(5 points)

(b) Clearly, $\|x\|_{\mathcal{W}} \geq 0$ for all $x \in \mathcal{W}$. If $\|x\|_{\mathcal{W}} = 0$, then $|x_k| w_k = 0$ for all $k \in \mathbb{N}$, which implies that $x_k = 0$ for all $k \in \mathbb{K}$ so that $x = 0$.
(2 points)

If $x \in \mathcal{W}$ and $\lambda \in \mathbb{K}$, then

$$\|\lambda x\|_{\mathcal{W}} = \sup_{k\in\mathbb{N}} |\lambda x_k| w_k = \sup_{k\in\mathbb{N}} |\lambda|\, |x_k| w_k = |\lambda| \sup_{k\in\mathbb{N}} |x_k| w_k = |\lambda| \|x\|_{\mathcal{W}}.$$

(4 points)

If $x, y \in \mathcal{W}$, then

$$\begin{aligned} \|x + y\|_{\mathcal{W}} &= \sup_{k\in\mathbb{N}} |x_k + y_k| w_k \\ &\leq \sup_{k\in\mathbb{N}} (|x_k| w_k + |y_k| w_k) \\ &\leq \sup_{k\in\mathbb{N}} |x_k| w_k + \sup_{k\in\mathbb{N}} |y_k| w_k \\ &= \|x\|_{\mathcal{W}} + \|y\|_{\mathcal{W}}. \end{aligned}$$

(4 points)

(c) *Version 1.* If $w_k = k$, then the norms $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_\infty$ are not equivalent. Indeed, the sequence

$$e_n = (0, \dots, 0, 1, 0, 0, 0, \dots),$$

where the 1 is at the n-th position, clearly lies in $\mathcal{W}$. We have that

$$\|e_n\|_\infty = 1 \quad \text{and} \quad \|e_n\|_{\mathcal{W}} = n.$$

Therefore, there does not exist a constant $C > 0$ such that $\|x\|_{\mathcal{W}} \leq C\|x\|_\infty$ for all $x \in \mathcal{W}$. This immediately implies that the norms are not equivalent.

Version 2. If $w_k = 1 + 1/k$, then the norms $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_\infty$ are equivalent. Indeed, since $1 \leq w_k \leq 2$ for all $k \in \mathbb{K}$ it easily follows that

$$\|x\|_\infty \leq \|x\|_{\mathcal{W}} \leq 2\|x\|_\infty$$

for all $x \in \mathcal{W}$, which means that the norms are equivalent.

Solution of problem 2, version 1 (5 + 10 + 15 = 30 points)

(a) For any $f \in \mathcal{C}([0,1],\mathbb{K})$ we have

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| = \sup_{x\in[0,1]} |f(1-x)| = \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty.$$

(3 points)

This implies that the operator norm of T is given by

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 1.$$

(2 points)

(b) We have that $Tf = f$ if and only if

$$f(1-x) = f(x) \quad \text{for all} \quad x \in [0,1],$$

which means that the graph of f is symmetric with respect to the vertical line $x = \frac{1}{2}$. Clearly, there exist nonzero functions f which satisfy this condition. A concrete example is given by any nonzero constant function. We conclude that $\lambda = 1$ is an eigenvalue of T.

(5 points)

We have that $Tf = -f$ if and only if

$$f(1-x) = -f(x) \quad \text{for all} \quad x \in [0,1],$$

which means that the graph of f is antisymmetric with respect to the vertical line $x = \frac{1}{2}$. Clearly, there exist nonzero functions f which satisfy this condition. A concrete example is given by $f(x) = x - \frac{1}{2}$. We conclude that $\lambda = -1$ is an eigenvalue of T.

(5 points)

(c) If $(T-\lambda)f = g$, then

$$f(1-x) - \lambda f(x) = g(x) \quad \text{for all} \quad x \in [0,1].$$

Replacing x by $1-x$ gives

$$f(x) - \lambda f(1-x) = g(1-x) \quad \text{for all} \quad x \in [0,1].$$

Hence, we obtain the following system of equations:

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} f(x) \\ f(1-x) \end{pmatrix} = \begin{pmatrix} g(x) \\ g(1-x) \end{pmatrix} \quad \text{for all} \quad x \in [0,1].$$

(5 points)

The coefficient matrix is invertible if and only if $\lambda \notin \{-1,1\}$. In that case we find that

$$f(x) = -\frac{1}{\lambda^2 - 1}(\lambda g(x) + g(1-x)).$$

(7 points)

This also implies that

$$(T - \lambda)^{-1} = -\frac{\lambda}{\lambda^2 - 1} I - \frac{1}{\lambda^2 - 1} T$$

is bounded because it is a linear combination of the bounded operators I and T. We conclude that $\lambda \in \rho(T)$.
(3 points)

Solution of problem 2, version 2 (5 + 10 + 10 + 5 = 30 points)

(a) For any $f \in \mathcal{C}([-\pi,\pi],\mathbb{K})$ we have

$$\|Tf\|_\infty = \sup_{x\in[-\pi,\pi]} |Tf(x)| = \sup_{x\in[-\pi,\pi]} |f(-x)| = \sup_{x\in[-\pi,\pi]} |f(x)| = \|f\|_\infty.$$

(3 points)

This implies that the operator norm of T is given by

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 1.$$

(2 points)

(b) We have that $Tf = f$ if and only if

$$f(-x) = f(x) \quad \text{for all} \quad x \in [-\pi,\pi],$$

which means that the graph of f is symmetric with respect to the y-axis. Clearly, there exist nonzero functions f which satisfy this condition. A concrete example is given by any nonzero constant function. We conclude that $\lambda = 1$ is an eigenvalue of T.
(5 points)

We have that $Tf = -f$ if and only if

$$f(-x) = -f(x) \quad \text{for all} \quad x \in [-\pi,\pi],$$

which means that the graph of f is antisymmetric with respect to the y-axis. Clearly, there exist nonzero functions f which satisfy this condition. A concrete example is given by $f(x) = x$. We conclude that $\lambda = -1$ is an eigenvalue of T.
(5 points)

(c) If $(T-\lambda)f = g$, then

$$f(-x) - \lambda f(x) = g(x) \quad \text{for all} \quad x \in [-\pi,\pi].$$

Replacing x by $-x$ gives

$$f(x) - \lambda f(-x) = g(-x) \quad \text{for all} \quad x \in [-\pi,\pi].$$

Hence, we obtain the following system of equations:

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} f(x) \\ f(-x) \end{pmatrix} = \begin{pmatrix} g(x) \\ g(-x) \end{pmatrix} \quad \text{for all} \quad x \in [-\pi,\pi].$$

(5 points)

The coefficient matrix is invertible if and only if $\lambda \notin \{-1,1\}$. In that case we find that

$$f(x) = -\frac{1}{\lambda^2-1}(\lambda g(x) + g(-x)).$$

(7 points)

This also implies that

$$(T - \lambda)^{-1} = -\frac{\lambda}{\lambda^2 - 1} I - \frac{1}{\lambda^2 - 1} T$$

is bounded because it is a linear combination of the bounded operators I and T. We conclude that $\lambda \in \rho(T)$.
(3 points)

Solution of problem 3, version 1 and 2 (5 + 5 + 5 = 15 points)

(a) Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$. The fact that $f_y : X \to \mathbb{C}$ defined by $f_y(x) = (Tx, y)/\|y\|$ is a linear map follows from:

$$\begin{aligned} f_y(\lambda x + \mu z) &= \frac{(T(\lambda x + \mu z), y)}{\|y\|} \\ &= \frac{(\lambda Tx + \mu Tz, y)}{\|y\|} \\ &= \frac{\lambda(Tx, y)}{\|y\|} + \frac{\mu(Tz, y)}{\|y\|} \\ &= \lambda f_y(x) + \mu f_y(z). \end{aligned}$$

(5 points)

(b) Let $x \in X$ be arbitrary, then

$$|f_y(x)| = \frac{|(Tx, y)|}{\|y\|} \leq \frac{\|Tx\|\,\|y\|}{\|y\|} = \|Tx\|.$$

This shows that

$$\sup_{y \neq 0} |f_y(x)| < \infty$$

for all $x \in X$.
(5 points)

(c) By the uniform boundedness principle it follows that $\sup_{y \neq 0} \|f_y\| < \infty$. Since $(Tx, y) = (x, Ty)$ it follows with $x = Ty/\|y\|$ that

$$\frac{\|Ty\|^2}{\|y\|^2} = f_y\left(\frac{Ty}{\|y\|}\right) \leq \|f_y\| \frac{\|Ty\|}{\|y\|}$$

so that

$$\|T\| = \sup_{y \neq 0} \frac{\|Ty\|}{\|y\|} \leq \sup_{y \neq 0} \|f_y\| < \infty$$

which shows that T is bounded.
(5 points)

Solution of problem 4, version 1 (8 points)

Method 1. Clearly, T is bounded. In addition, T is injective: indeed, $Tx = 0$ implies $x = 0$. If $\operatorname{ran} T$ is closed in ℓ^∞, then $T : \ell^\infty \to \operatorname{ran} T$ is a bijective operator between the Banach spaces ℓ^∞ and $\operatorname{ran} T$. A corollary of the Open Mapping Theorem then implies that the inverse $T^{-1} : \operatorname{ran} T \to \ell^\infty$ is bounded.
(3 points)

The inverse of $T : \ell^\infty \to \operatorname{ran} T$ is given by:

$$S : \operatorname{ran} T \to \ell^\infty, \quad S(x_1, x_2, x_3, x_4, \dots) = (x_1, 2x_2, 3x_3, 4x_4, \dots).$$

Indeed, $ST = I_{\ell^\infty}$ and $TS = I_{\operatorname{ran} T}$ and since inverses are unique it follows that S must be the inverse of T.
(2 points)

However, S is *not* bounded since for the unit vector $e_n = (0, \dots, 0, 1, 0, 0, 0, \dots)$ it follows that $\|e_n\|_\infty = 1$ while $\|Se_n\|_\infty = n \to \infty$. From this contradiction we conclude that $\operatorname{ran} T$ cannot be closed in ℓ^∞.
(3 points)

Method 2. Clearly, T is bounded. In addition, T is injective: indeed, $Tx = 0$ implies $x = 0$. If $\operatorname{ran} T$ is closed in ℓ^∞, then we can apply the "Closed Range Proposition" (which is in fact an application of the Open Mapping Theorem): there exists a constant $c > 0$ such that

$$\|Tx\| \geq c\|x\| \quad \text{for all} \quad x \in \ell^\infty.$$

(5 points)

However, for the unit vectors $e_n = (0, \dots, 0, 1, 0, 0, 0, \dots)$ we have $\|e_n\|_\infty = 1$ while $\|Te_n\|_\infty = 1/n \to 0$. This implies that the inequality above cannot hold. From this contradiction we conclude that $\operatorname{ran} T$ cannot be closed in ℓ^∞.
(3 points)

Solution of problem 4, version 2 (8 points)

Method 1. Clearly, T is bounded. In addition, T is injective: indeed, $Tx = 0$ implies $x = 0$. If $\operatorname{ran} T$ is closed in ℓ^1, then $T : \ell^1 \to \operatorname{ran} T$ is a bijective operator between the Banach spaces ℓ^1 and $\operatorname{ran} T$. A corollary of the Open Mapping Theorem then implies that the inverse $T^{-1} : \operatorname{ran} T \to \ell^1$ is bounded.
(3 points)

The inverse of $T : \ell^1 \to \operatorname{ran} T$ is given by:

$$S : \operatorname{ran} T \to \ell^1, \quad S(x_1, x_2, x_3, x_4, \dots) = (x_1, 4x_2, 9x_3, 16x_4, \dots).$$

Indeed, $ST = I_{\ell^1}$ and $TS = I_{\operatorname{ran} T}$ and since inverses are unique it follows that S must be the inverse of T.
(2 points)

However, S is *not* bounded since for the unit vector $e_n = (0, \dots, 0, 1, 0, 0, 0, \dots)$ it follows that $\|e_n\|_1 = 1$ while $\|Se_n\|_1 = n^2 \to \infty$. From this contradiction we conclude that $\operatorname{ran} T$ cannot be closed in ℓ^1.
(3 points)

Method 2. Clearly, T is bounded. In addition, T is injective: indeed, $Tx = 0$ implies $x = 0$. If $\operatorname{ran} T$ is closed in ℓ^1, then we can apply the "Closed Range Proposition" (which is in fact an application of the Open Mapping Theorem): there exists a constant $c > 0$ such that

$$\|Tx\| \geq c\|x\| \quad \text{for all} \quad x \in \ell^1.$$

(5 points)

However, for the unit vectors $e_n = (0, \dots, 0, 1, 0, 0, 0, \dots)$ we have $\|e_n\|_1 = 1$ while $\|Te_n\|_1 = 1/n^2 \to 0$. This implies that the inequality above cannot hold. From this contradiction we conclude that $\operatorname{ran} T$ cannot be closed in ℓ^1.
(3 points)

Solution of problem 5, version 1 (4 + 4 + 4 = 12 points)

(a) We have that

$$|f(x)| = |a_1x_1 + a_2x_2| \leq |a_1|\,|x_1| + |a_2|\,|x_2| \leq \max\{|a_1|, |a_2|\}\|x\|_1.$$

(2 points)

For the vectors $x = (1,0)$ and $y = (0,1)$ we have $\|x\|_1 = \|y\|_1 = 1$ while $|f(x)| = |a_1|$ and $|f(y)| = |a_2|$. Hence, we conclude that

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|_1} = \max\{|a_1|, |a_2|\}.$$

(2 points)

(b) If $x = (1,0)$ then $f(x) = g(x)$ implies that $a_1 = 7$. With this choice for a_1 it follows by linearity that $f(x) = g(x)$ for all $x \in V$.
(2 points)

By part (a) it follows that $\|g\| = 7$. So in order to have $\|f\| = 7$ as well, we must have that $|a_2| \leq 7$, or, equivalently, $-7 \leq a_2 \leq 7$.
(2 points)

(c) We can see the map f, where a_1 and a_2 are as in part (b), as a norm preserving extension of the map g restricted to V. This implies that norm preserving extensions, of which the *existence* is guaranteed by the Hahn-Banach Theorem, need not be unique.
(4 points)

Solution of problem 5, version 2 (4 + 4 + 4 = 12 points)

(a) We have that

$$|f(x)| = |a_1x_1 + a_2x_2| \leq |a_1|\,|x_1| + |a_2|\,|x_2| \leq (|a_1| + |a_2|)\|x\|_\infty.$$

(2 points)

For the vectors $x = (1, 1)$ we have $\|x\|_\infty = 1$ while $|f(x)| = |a_1| + |a_2|$. Hence, we conclude that

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|_\infty} = |a_1| + |a_2|.$$

(2 points)

(b) If $x = (1, 1)$ then $f(x) = g(x)$ implies that $a_1 + a_2 = 8$, or, equivalently, $a_2 = 8-a_1$. With this choice for a_1 and a_2 it follows by linearity that $f(x) = g(x)$ for all $x \in V$.
(2 points)

By part (a) it follows that $\|g\| = 8$. So in order to have $\|f\| = 8$ as well, we must have that $|a_1| + |8 - a_1| = 8$, or, equivalently, $0 \leq a_1 \leq 8$.
(2 points)

(c) We can see the map f, where a_1 and a_2 are as in part (b), as a norm preserving extension of the map g restricted to V. This implies that norm preserving extensions, of which the *existence* is guaranteed by the Hahn-Banach Theorem, need not be unique.
(4 points)